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Padé Approximants as Limits of Best Rational Approximants

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1. INTRODUCTION

In 1934 J. L. Walsh noted in [4] that the Taylor polynomial $\sum_{k=0}^{n} a_k z^k$ of an analytic function f could be obtained by taking the limit as $\epsilon \downarrow 0$ of the best (Chebyshev) nth degree polynomial approximant of degree $\geqslant N$ to f in the disk $|z| \leqslant \epsilon$. Later he generalized this result to Padé approximants of analytic functions [5]; finally in [6] he proved the following

THEOREM. *Let $f(x) \equiv a_0 + a_1 x + \cdots + a_{m+n} x^{m+n} + 0(x^{m+n+1})$, $m \geqslant 0$, $n \geqslant 0$, be of class $C^{m+n+1}[0, \delta]$ for some $\delta > 0$. Let $R_\epsilon = P_\epsilon / Q_\epsilon$ denote a rational function of type (m, n) which best approximates f in the Chebyshev sense on $[0, \epsilon]$. Suppose further that the determinant*

$$\begin{vmatrix} a_m & a_{m-1} & \cdots & a_{m-n+1} \\ a_{m+1} & a_m & \cdots & a_{m-n+2} \\ & & \cdot\ \cdot\ \cdot & \\ a_{m+n-1} & a_{m+n-2} & \cdots & a_m \end{vmatrix} \neq 0, \qquad \text{and} \quad a_0 \neq 0. \tag{*}$$

Here $a_j = 0$ if $j < 0$. Then as $\epsilon \downarrow 0$, R_ϵ approaches the $[m/n]$ Padé approximant $R_{m,n}$ of f on any closed set where $R_{m,n}$ is analytic.

In this paper we show that the Padé approximant to *any* function $f \in C^{m+n+1}[0, \delta]$ is obtained by taking the best rational approximant on the

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interval $[0, \epsilon]$, and then letting $\epsilon \downarrow 0$. Our main contribution is an approixi-mation theoretic proof of this fact without assuming (*).

We require that the functions with which we deal be real-valued since the proofs rely on real variable techniques. When referring to the degree of a polynomial $P(\deg P)$ we will mean its exact degree (the polynomial 0 has degree -1). If P and Q are polynomials, P/Q is defined continuously at the removable singular points.

2. Main Result

Let $f \in C^{m+n+1}[0, \delta]$ $(m \geqslant 0, n \geqslant 0)$ for some δ, $0 < \delta \leqslant 1$. We set

$$\begin{aligned} f(x) &= T_{m+n}(x) + r_{m+n}(x) \\ &= a_0 + a_1 x + \cdots + a_{m+n} x^{m+n} + r_{m+n}(x), \end{aligned} \tag{1}$$

where $a_j = f^{(j)}(0)/j!$, $j = 0,..., m+n$. The $[m/n]$ Padé approximant of f is the unique rational function $R_{m,n}$ of the form P_m/Q_n, where P_m and Q_n are polynomials of degree no greater than m and n, respectively, such that $f(x)\, Q_n(x) - P_m(x) = 0(x^{m+n+1})$ as $x \downarrow 0$. For the uniqueness of $R_{m,n}$ and other details, cf. [3]. From this definition, the following lemma follows immediately.

Lemma 1. *Let P_m and Q_n be polynomials of degree no greater than m and n, respectively, $Q_n \not\equiv 0$. Then $R_{m,n} = P_m/Q_n$ is the $[m/n]$ Padé approximant of f if and only if P_m and Q_n satisfy the following equations*:

$$(fQ_n - P_m)^{(j)}(0) = 0, \qquad j = 0,..., m+n. \tag{2}$$

We call (2) the Padé equations. Let $\mathscr{R} \equiv \mathscr{R}_{m,n}$ be the collection of all rational functions P/Q where P and Q are polynomials of degree no greater than m and n, respectively. For $0 < \epsilon \leqslant \delta$, let $R_\epsilon \in \mathscr{R}$ satisfy

$$\| f - R_\epsilon \|_{[0,\epsilon]} = \min_{g \in \mathscr{R}} \| f - g \|_{[0,\epsilon]}, \tag{3}$$

where $\| \ \|_{[0,\epsilon]}$ is the supremum norm over the interval $[0, \epsilon]$. We will establish the following theorem using the elements of the theory of best approximation.

Theorem 1. *Let $R_{m,n}$ be the $[m/n]$ Padé approximant of f. For each ϵ, $0 < \epsilon \leqslant \delta$, let $R_\epsilon \in \mathscr{R}$ satisfy* (3). *Then there is a real neighborhood Ω of* 0 *such that $R_\epsilon \to R_{m,n}$ uniformly on Ω, as $\epsilon \downarrow 0$.*

To prove this theorem, we need the following standard results in approximation theory. See, for instance, Cheney [1, p. 163], and Lorentz [2, p. 40].

LEMMA 2. *In order that an irreducible rational function $R_\epsilon = P_\epsilon/Q_\epsilon$ be the best approximation to f from $\mathscr{R}$ on* $[0, \epsilon]$, *it is necessary and sufficient that the error $f - R_\epsilon$ will have at least* $2 + \max\{m + \deg Q_\epsilon, n + \deg P_\epsilon\}$ *alternations.*

LEMMA 3. (Markov). *Let p_N be a polynomial of degree no greater than $N(\geqslant 0)$ such that $|p_N(x)| \leqslant M$ for all x on* $[0, \epsilon]$, $\epsilon > 0$. *Then $|p_N'(x)| \leqslant 2N^2M/\epsilon$ for all x on* $[0, \epsilon]$

We now begin to prove Theorem 1. Write R_ϵ in an irreducible form $R_\epsilon = P_\epsilon/Q_\epsilon$, where $P_\epsilon(x) = \sum_{j=0}^m b_{\epsilon,j}x^j$, $Q_\epsilon(x) = \sum_{j=0}^n c_{\epsilon,j}x^j$, and $\sum_{j=0}^n |c_{\epsilon,j}| = 1$. Let $\nu(\epsilon) = \max\{m + \deg Q_\epsilon, n + \deg P_\epsilon\}$. By Lemma 2, $f - R_\epsilon$ has at least $\nu(\epsilon) + 1$ (distinct) zeros in $(0, \epsilon)$. Let $\tilde{P}_\epsilon(x) = P_\epsilon(x)\, x^{m+n-\nu(\epsilon)}$, and $\tilde{Q}_\epsilon(x) = Q_\epsilon(x)\, x^{m+n-\nu(\epsilon)}$. Then counting multiplicities, $f\tilde{Q}_\epsilon - \tilde{P}_\epsilon$ has at least $m + n + 1$ zeros in $[0, \epsilon]$. By Rolle's Theorem we have

$$(f\tilde{Q}_\epsilon - \tilde{P}_\epsilon)^{(j)}(\xi_{\epsilon,j}) = 0, \qquad j = 0,\dots, m + n, \tag{4}$$

where $\xi_{\epsilon,0},\dots, \xi_{\epsilon,m+n}$ lie in $[0, \epsilon]$. Furthermore, it is easy to see that $\deg \tilde{P}_\epsilon \leqslant m$ and $\deg \tilde{Q}_\epsilon \leqslant n$. By the normalization of Q_ϵ, we have $|c_{\epsilon,j}| \leqslant 1$, so that there is a sequence $\epsilon_k \downarrow 0$ such that for all k, $m + n - \nu(\epsilon_k) = l$, a nonnegative integer, and $c_{\epsilon_k,j}$ converges to some c_j for $j = 0,\dots, n$. (Observe that ϵ_k can be chosen as a subsequence of any given sequence of positive numbers converging to 0.) Define $Q(x) = \sum_{j=0}^n c_jx^j$, and $\tilde{Q}(x) = x^lQ(x)$. Clearly, $\deg \tilde{Q} \leqslant n$, and it is obvious that $\sum |c_j| = 1$ so that $\tilde{Q} \not\equiv 0$. Hence, since $\xi_{\epsilon_k,j} \to 0$ for $j = 0,\dots, m + n$, we have $(f\tilde{Q}_{\epsilon_k})^{(j)}(\xi_{\epsilon_k,j}) \to (f\tilde{Q})^{(j)}(0)$, $0 \leqslant j \leqslant m + n$. But since $\deg \tilde{P}_\epsilon \leqslant m$, $(f\tilde{Q}_\epsilon)^{(j)}(\xi_{\epsilon,j}) = 0$ whenever $m < j \leqslant m + n$, from (4). Hence, we have

$$(f\tilde{Q})^{(j)}(0) = 0, \qquad j = m + 1,\dots, m + n. \tag{5}$$

These are "half" of the Padé equations (2). To obtain the other "half", note that

$$\|f - R_\epsilon\|_{[0,\epsilon]} = \|f - \tilde{P}_\epsilon/\tilde{Q}_\epsilon\|_{[0,\epsilon]} \leqslant \|f - T_m\|_{[0,\epsilon]},$$

where $T_m(x) = a_0 + \cdots + a_mx^m$. Hence, it follows that

$$\begin{aligned}\|T_m\tilde{Q}_\epsilon - \tilde{P}_\epsilon\|_{[0,\epsilon]} &\leqslant \|f\tilde{Q}_\epsilon - \tilde{P}_\epsilon\|_{[0,\epsilon]} + \|(f - T_m)\tilde{Q}_\epsilon\|_{[0,\epsilon]}\\ &\leqslant \|f - R_\epsilon\|_{[0,\epsilon]}\|\tilde{Q}_\epsilon\|_{[0,\epsilon]} + \|f - T_m\|_{[0,\epsilon]}\|\tilde{Q}_\epsilon\|_{[0,\epsilon]}\\ &\leqslant 2\|\tilde{Q}_\epsilon\|_{[0,\epsilon]}\|f - T_m\|_{[0,\epsilon]} \leqslant 2\|f - T_m\|_{[0,\epsilon]}.\end{aligned}$$

By the definition of T_m, it is clear that $\|f - T_m\|_{[0,\epsilon]} = 0(\epsilon^{m+1})$. Thus, by Lemma 3, we have

$$\|(T_m\tilde{Q}_\epsilon - \tilde{P}_\epsilon)^{(j)}\|_{[0,\epsilon]} = 0(\epsilon^{m-j+1}), \qquad j = 0,\dots, m. \tag{6}$$

In particular, for $j = 0,..., m$,

$$\| \tilde{P}_\epsilon^{(j)} \|_{[0,\epsilon]} \leqslant \|(T_m\tilde{Q}_\epsilon)^{(j)} \|_{[0,\epsilon]} + 0(\epsilon^{m-j+1})$$
$$\leqslant \|(T_m\tilde{Q}_\epsilon)^{(j)} \|_{[0,1]} + 0(\epsilon^{m-j+1}) = 0(1). \qquad (7)$$

In (7), if we put $j = m, j = m - 1,...,$ and $j = 0$, consecutively, we see that the sequences $\{b_{\epsilon_k,j}\}$, $j = 0,..., m$, are bounded and hence have convergent subsequences $\{b_{\epsilon_k',j}\}$. Let $b_j = \lim b_{\epsilon_k',j}$, $P(x) = \sum_{j=0}^{m} b_j x^j$, and $\tilde{P}(x) = x^l P(x)$. Clearly, $\deg \tilde{P} \leqslant m$. From (4) we conclude that

$$(f\tilde{Q} - \tilde{P})^{(j)}(0) = 0, \qquad j = 0,..., m. \qquad (8)$$

These are the other "half" of the Padé equations (2). Combining (5) and (8), we have

$$(f\tilde{Q} - \tilde{P})^{(j)}(0) = 0, \qquad j = 0,..., m + n.$$

By Lemma 1 we know that $R_{m,n} = \tilde{P}/\tilde{Q}$ is the $[m/n]$ Padé approximant of f. It follows from the uniqueness property of the Padé approximant that as $k \to \infty$, $R_{\epsilon_k} \to R_{m,n}$, coefficientwise. Hence, $R_\epsilon \to R_{m,n}$ coefficientwise, as $\epsilon \downarrow 0$. Taking Ω to be any open interval $(-\eta, \eta)$ for which $[-\eta, \eta]$ contains no pole of $R_{m,n}$, it follows that $R_\epsilon \to R_{m,n}$ uniformly on Ω, as $\epsilon \downarrow 0$.

References

1. E. W. Cheney, "Introduction to Approximation Theory," McGraw-Hill, New York, 1966.
2. G. G. Lorentz, "Approximation of Functions," Holt, New York, 1966.
3. H. W. Wall, "Analytic Theory of Continued Fractions," Van Nostrand, Princeton, NJ, 1948.
4. J. L. Walsh, On approximation to an analytic function by rational functions of best approximation, *Math. Z.* **38** (1934), 163–176.
5. J. L. Walsh, Padé approximants as limits of rational functions of best approximation, *J. Math. Mech.* **13** (1964), 305–312.
6. J. L. Walsh, Padé approximants as limits of rational functions of best approximation, real domain, *J. Approximation Theory* **11** (1974), 225–230.